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ADDENDUM

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Abstract. The perturbed ground-state eigenvalues for the harmonic oscillator with negative quartic and with cubic energy distortion are calculated by numerical integration. These agree with the author's results using an asymptotic series.

Drummond (1981) has produced a table of complex eigenvalues for five energy levels for the negative quartic and six energy levels for cubic distortion of the harmonic oscillator. He did this by using an interpretation of the asymptotic series for the energy, so it is considered worthwhile to verify these results using an independent calculation. The method used here is a direct integration of the Schrödinger wave equation.

The normalised Schrödinger equation for the anharmonic oscillator is

$$(-\mathrm{d}^2/\mathrm{d}x^2+x^2-\lambda x^4)\psi=(E+\mathrm{i}\varepsilon)\psi$$

and $(E+\mathrm{i}\varepsilon)$ is an eigenvalue if ψ represents a standing wave near the origin decaying through the potential wall and becoming a decaying and outgoing progressive wave for large x.

To search for the eigenvalue, let $\psi=u+\mathrm{i}v$, approximately normalise the wave period with

$$1+3\lambda^{1/4}t=(1+\lambda^{1/4}x)^3, \qquad \text{let } \bar{u}=\mathrm{d}u/\mathrm{d}t, \qquad \bar{v}=\mathrm{d}v/\mathrm{d}t$$

and do four Runge–Kutta integrations with steps of 0.01 for four values of $(E+\mathrm{i}\varepsilon)$ forming a rectangular search grid around the initial estimate of the eigenvalue. If ω is an approximation to the frequency then $\bar{u}/\omega v$ and $\bar{v}/\omega u$ are close to ± 1 for large t for a progressive wave. This is used to estimate the eigenvalue by interpolation. These estimates of $(E+\mathrm{i}\varepsilon)$ lie on a roughly circular spiral which decays approximately as $(1/t)$, has a period equal to half the wave period and has two spikes when $\bar{u}$ and v or $\bar{v}$ and u are simultaneously very small. If the initial search grid is too large or off-centre, the two halves of the spiked circle degenerate to a line with a hump in it and a line with a loop in it. When the search grid is close to the spiral the arithmetical accuracy depends mainly on the size of the spiral.

The values of E and ε at 7 points on the smooth part of the spiral for t near 100, 200 and 400 were extrapolated geometrically to infinite t.

The resulting extrapolations are listed in table 1.

Table 1. Complex eigenvalues of the ground state for the nonlinear oscillator with a negative distortion term λx^4.

	$E+\mathrm{i}\varepsilon$			
	From series		From integral	
λ	Real	Imaginary	Real	Imaginary
0.01	0.992 363 2206			
0.02	0.984 427 6698			
0.03	0.976 146 1974	0.000 000 0027		
0.04	0.967 451 234	0.000 000 596	0.967 451 24	0.000 000 60
0.05	0.958 2336	0.000 014 6	0.958 233 64	0.000 014 56
0.06	0.948 330	0.000 119	0.948 329 8	0.000 119 1
0.07	0.937 582	0.000 521	0.937 581	0.000 521
0.08	0.925 95	0.001 54	0.925 942	0.001 544
0.09	0.913 55	0.003 49	0.913 548	0.003 521
0.10	0.900 6	0.006 6	0.900 67	0.006 69
0.12	0.874 6	0.016 5	0.874 80	0.016 87
0.15	0.839	0.039	0.839 40	0.040 11
0.20	0.793	0.09	0.794 9	0.089 4
0.25	0.76	0.14	0.765 9	0.141 0
0.3	0.74	0.19	0.747 5	0.190 1
0.4	0.72	0.27	0.728 8	0.277 3
0.5	0.72	0.35	0.722 9	0.351 5
0.6	0.72	0.42	0.723 4	0.415 6
0.7	0.72	0.48	0.727 2	0.472 0
0.8	0.73	0.53	0.733 1	0.522 4
0.9	0.74	0.58	0.740 1	0.568 1
1.0	0.75	0.62	0.747 7	0.610 0

Table 2. The complex eigenvalue of the ground state for the nonlinear oscillator with an energy distortion term λx^3.

	$E+\mathrm{i}\varepsilon$			
	From series		From integral	
λ	Real	Imaginary	Real	Imaginary
0.01	0.999 931 231 826	0		
—	—			
0.15	0.983 476 9			
0.2	0.968 63	0.000 02	0.968 632 6	0.000 016 1
0.25	0.944 8	0.001 4	0.944 80	0.001 40
0.3	0.910	0.013	0.909 69	0.013 09
0.4	0.85	0.08	0.847 8	0.084 9
0.5	0.82	0.18	0.827 7	0.178 1
0.6	0.83	0.27	0.835 0	0.263 8
0.7	0.85	0.34	0.856	0.338
0.8	0.9	0.4	0.883	0.401
0.9	0.9	0.5	0.913	0.457
1.0	1	0.5	0.944	0.506

These are compared with the results from the asymptotic series. The integration error was estimated to be 10^{-9} so the asymptotic series is better than the integration for very small λ. For larger λ the self-consistency error in extrapolating from the spiral is less than one in the last digit quoted.

Apart from a transcription error for $\lambda = 0.05$ the values derived from the asymptotic series all agree with the new results within 1 or 2 in the last digit quoted and serve as excellent initial estimates in searching for more accurate values.

For the cubic distortion $(-\lambda x^3)$ the integration was carried out from $x = -10$ to $t = 50$, 100, 200 in steps of 0.05, then interpolated and extrapolated as before. The arithmetic error was estimated to be 10^{-7}. The results are listed in table 2. Again the asymptotic series is better for very small λ and consistent for larger λ.

Reference

Drummond J E 1981 *J. Phys. A: Math. Gen.* **14** 1651–61